A calculation for the error of muon pair invariant mass $\sigma(M_{\mu^+\mu^-})$

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Abstract

In this note we will discuss the calculation of error of reconstructed muon pair mass $\sigma(M_{\mu^+\mu^-})$, which will be used in $h \to \mu^+\mu^-$ analysis. First, we will show the case of using muon pair only. Next, we will discuss the error including FSR photon correction.

1 Calculation of error of muon pair mass

Define a function $f(x_1 \ldots x_n)$, variables $x_1 \ldots x_n$, variance $\sigma_{x_1} \ldots \sigma_{x_n}$, and covariance like $\sigma_{x_1x_2}$. The error Δf can be written as;

$$(\Delta f)^2 = \sum_i \sum_j \frac{\partial f}{\partial x_i} V_{ij} \frac{\partial f}{\partial x_j}, \tag{1}$$

where V is the covariance matrix.

Here, we will discuss the muon pair invariant mass $M_{\mu^+\mu^-} \equiv M$ and its error $\sigma(M_{\mu^+\mu^-}) \equiv \sigma(M)$. A four-momentum of single muon can be written like $(E_1 \quad p_{1x} \quad p_{1y} \quad p_{1z})$. The M can be written as;

$$M = \sqrt{(E_1 + E_2)^2 - (p_{1x} + p_{2x})^2 - (p_{1y} + p_{2y})^2 - (p_{1z} + p_{2z})^2}, \tag{2}$$

and the error $\sigma(M)$ is given by;

$$\sigma^2(M) = \sum_{i=E_1,p_{1x},p_{1y},p_{1z},E_2,p_{2x},p_{2y},p_{2z}} \sum_{j=E_1,p_{1x},p_{1y},p_{1z},E_2,p_{2x},p_{2y},p_{2z}} \frac{\partial M}{\partial x_i} V_{ij} \frac{\partial M}{\partial x_j}. \tag{3}$$

This equation contains 64 terms in total.

First, we will perform partial derivative. We will calculate $\frac{\partial M}{\partial E_1}$ and $\frac{\partial M}{\partial p_{1x}}$ as the example. We have;

$$\frac{\partial M}{\partial E_1} = \frac{E_1 + E_2}{M} \tag{4}$$

$$\frac{\partial M}{\partial p_{1x}} = -\frac{p_{1x} + p_{2x}}{M} \tag{5}$$

Next, we will consider the covariance. It is clear that there should be some correlation among the component of four momentum of single muon, because we can calculate the invariant mass using four momentum. We must consider the covariance like $\sigma_{E_1p_{1x}}$. However, the correlation between two muons would be negligible. For example, it is hard to estimate the correlation between E_1 and p_{2x}. Therefore, we can ignore half of terms in (3). In final, we have;

$$\begin{aligned}\sigma^2(M) = \frac{1}{M^2} [\, & E_2\sigma^2_{E_1}E_2 - E_2\sigma_{E_1p_{1x}}p_{2x} - E_2\sigma_{E_1p_{1y}}p_{2y} - E_2\sigma_{E_1p_{1z}}p_{2z} \\ & - p_{2x}\sigma_{p_{1x}E_1}E_2 + p_{2x}\sigma^2_{p_{1x}}p_{2x} + p_{2x}\sigma_{p_{1x}p_{1y}}p_{2y} + p_{2x}\sigma_{p_{1x}p_{1z}}p_{2z} \\ & - p_{2y}\sigma_{p_{1y}E_1}E_2 + p_{2y}\sigma_{p_{1y}p_{1x}}p_{2x} + p_{2y}\sigma^2_{p_{1y}}p_{2y} + p_{2y}\sigma_{p_{1y}p_{1z}}p_{2z}\end{aligned}$$

$$
\begin{aligned}
&- p_{2z}\sigma_{p_{1z}E_1}E_2 + p_{2z}\sigma_{p_{1z}p_{1x}}p_{2x} + p_{2z}\sigma_{p_{1z}p_{1y}}p_{2y} + p_{2z}\sigma^2_{p_{1z}}p_{2z} \\
&+ E_1\sigma^2_{E_2}E_1 - E_1\sigma_{E_2p_{2x}}p_{1x} - E_1\sigma_{E_2p_{2y}}p_{1y} - E_1\sigma_{E_2p_{2z}}p_{1z} \\
&- p_{1x}\sigma_{p_{2x}E_2}E_1 + p_{1x}\sigma^2_{p_{2x}}p_{1x} + p_{1x}\sigma_{p_{2x}p_{2y}}p_{1y} + p_{1x}\sigma_{p_{2x}p_{2z}}p_{1z} \\
&- p_{1y}\sigma_{p_{2y}E_2}E_1 + p_{1y}\sigma_{p_{2y}p_{2x}}p_{1x} + p_{1y}\sigma^2_{p_{2y}}p_{1y} + p_{1y}\sigma_{p_{2y}p_{2z}}p_{1z} \\
&- p_{1z}\sigma_{p_{2z}E_2}E_1 + p_{1z}\sigma_{p_{2z}p_{2x}}p_{1x} + p_{1z}\sigma_{p_{2z}p_{2y}}p_{1y} + p_{1z}\sigma^2_{p_{2z}}p_{1z} \;] \,.
\end{aligned} \tag{6}
$$

Now, define the matrix P and the covariance matrix Σ as;

$$
P_i \equiv \begin{pmatrix} E_i \\ p_{ix} \\ p_{iy} \\ p_{iz} \end{pmatrix}, \tag{7}
$$

$$
P_i^T \equiv \begin{pmatrix} E_i & -p_{ix} & -p_{iy} & -p_{iz} \end{pmatrix}, \tag{8}
$$

$$
\Sigma_i \equiv \begin{pmatrix} \sigma^2_{E_i} & \sigma_{E_ip_{ix}} & \sigma_{E_ip_{iy}} & \sigma_{E_ip_{iz}} \\ \sigma_{p_{ix}E_i} & \sigma^2_{p_{ix}} & \sigma_{p_{ix}p_{iy}} & \sigma_{p_{ix}p_{iz}} \\ \sigma_{p_{iy}E_i} & \sigma_{p_{iy}p_{ix}} & \sigma^2_{p_{iy}} & \sigma_{p_{iy}p_{iz}} \\ \sigma_{p_{iz}E_i} & \sigma_{p_{iz}p_{ix}} & \sigma_{p_{iz}p_{iy}} & \sigma^2_{p_{iz}} \end{pmatrix}. \tag{9}
$$

Using these definitions, we finally have;

$$
\sigma^2(M) = \frac{1}{M^2}\left[P_1^T\Sigma_2P_1 + P_2^T\Sigma_1P_2\right]. \tag{10}
$$

2 Error calculation of $\sigma(M)$ with considering FSR photon correction

In this section, we will discuss the invariant mass of muon pair corrected with FSR photons $M_{\mu^+\mu^-} \equiv M$, and its error $\sigma(M_{\mu^+\mu^-}) \equiv \sigma(M)$. The M can be written as;

$$
M = \sqrt{(E_{\mu1} + E_{\mu2} + E_{\gamma1} + E_{\gamma2})^2 - \sum_{\alpha=x,y,z}(p_{\mu1\alpha} + p_{\mu2\alpha} + p_{\gamma1\alpha} + p_{\gamma2\alpha})^2}, \tag{11}
$$

where $\gamma1(\gamma2)$ is the photon used for correction to $\mu1(\mu2)$. If we continue this calculation, we have;

$$
\begin{aligned}
M = [& \\
& M^2_{\mu1} + M^2_{\mu2} + M^2_{\gamma1} + M^2_{\gamma2} \\
& + 2(E_{\mu1}E_{\mu2} + E_{\mu1}E_{\gamma1} + E_{\mu1}E_{\gamma2} + E_{\mu2}E_{\gamma1} + E_{\mu2}E_{\gamma2} + E_{\gamma1}E_{\gamma2}) \\
& - 2\sum_{\alpha=x,y,z}(p_{\mu1\alpha}p_{\mu2\alpha} + p_{\mu1\alpha}p_{\gamma1\alpha} + p_{\mu1\alpha}p_{\gamma2\alpha} + p_{\mu2\alpha}p_{\gamma1\alpha} + p_{\mu2\alpha}p_{\gamma2\alpha} + p_{\gamma1\alpha}p_{\gamma2\alpha}) \\
&]^{1/2}.
\end{aligned} \tag{12}
$$

We know that $M_{\gamma1} = M_{\gamma2} = 0$ GeV and $M_h \simeq 125$ GeV >> $M_{\mu1} = M_{\mu2} \simeq 140$ MeV. Finally we have;

$$
\begin{aligned}
M = [& \\
& 2(E_{\mu1}E_{\mu2} + E_{\mu1}E_{\gamma1} + E_{\mu1}E_{\gamma2} + E_{\mu2}E_{\gamma1} + E_{\mu2}E_{\gamma2} + E_{\gamma1}E_{\gamma2}) \\
& - 2\sum_{\alpha=x,y,z}(p_{\mu1\alpha}p_{\mu2\alpha} + p_{\mu1\alpha}p_{\gamma1\alpha} + p_{\mu1\alpha}p_{\gamma2\alpha} + p_{\mu2\alpha}p_{\gamma1\alpha} + p_{\mu2\alpha}p_{\gamma2\alpha} + p_{\gamma1\alpha}p_{\gamma2\alpha}) \\
&]^{1/2}.
\end{aligned} \tag{13}
$$

The $\sigma(M)$ is given by;

$$\sigma^2(M) = \sum_{\substack{i=E_{\mu1},p_{\mu1x},p_{\mu1y},p_{\mu1z},\\ E_{\mu2},p_{\mu2x},p_{\mu2y},p_{\mu2z},\\ E_{\gamma1},p_{\gamma1x},p_{\gamma1y},p_{\gamma1z},\\ E_{\gamma2},p_{\gamma2x},p_{\gamma2y},p_{\gamma2z}}} \sum_{\substack{j=E_{\mu1},p_{\mu1x},p_{\mu1y},p_{\mu1z},\\ E_{\mu2},p_{\mu2x},p_{\mu2y},p_{\mu2z},\\ E_{\gamma1},p_{\gamma1x},p_{\gamma1y},p_{\gamma1z},\\ E_{\gamma2},p_{\gamma2x},p_{\gamma2y},p_{\gamma2z}}} \frac{\partial M}{\partial x_i} V_{ij} \frac{\partial M}{\partial x_j}. \tag{14}$$

This equation contains 256 terms in total.

As the first step, we will perform the partial derivative. We will consider $\frac{\partial M}{\partial E_{\mu1}}$ and $\frac{\partial M}{\partial p_{\mu1x}}$ as the examples. We have;

$$\frac{\partial M}{\partial E_{\mu1}} = \frac{E_{\mu2} + E_{\gamma1} + E_{\gamma2}}{M}, \tag{15}$$

$$\frac{\partial M}{\partial p_{\mu1x}} = -\frac{p_{\mu2x} + p_{\gamma1x} + p_{\gamma2x}}{M}. \tag{16}$$

Then, considering the correlation and covariance among variables. We can separate them into 2 categories as;

- correlation should be considered: $\sigma^2_{\mu1}$, $\sigma^2_{\mu2}$, $\sigma^2_{\gamma1}$, $\sigma^2_{\gamma2}$,
- not necessary to consider covariance because of less correlation: $\sigma_{\mu1\mu2}$, $\sigma_{\mu1\gamma2}$, $\sigma_{\mu2\gamma1}$, $\sigma_{\gamma1\gamma2}$, $\sigma_{\mu1\gamma1}$, $\sigma_{\mu2\gamma2}$.

From these, we can understand that only 64 terms are should be calculated. Finally we have;

$$\begin{array}{ll}
\sigma^2(M) & \\
= \frac{1}{M^2}\,[& \\
+ (E_{\mu2} + E_{\gamma1} + E_{\gamma2})\sigma^2_{E_{\mu1}}(E_{\mu2} + E_{\gamma1} + E_{\gamma2}) & -(E_{\mu2} + E_{\gamma1} + E_{\gamma2})\sigma_{E_{\mu1}p_{\mu1x}}(p_{\mu2x} + p_{\gamma1x} + p_{\gamma2x}) \\
- (E_{\mu2} + E_{\gamma1} + E_{\gamma2})\sigma_{E_{\mu1}p_{\mu1y}}(p_{\mu2y} + p_{\gamma1y} + p_{\gamma2y}) & -(E_{\mu2} + E_{\gamma1} + E_{\gamma2})\sigma_{E_{\mu1}p_{\mu1z}}(p_{\mu2z} + p_{\gamma1z} + p_{\gamma2z}) \\
- (p_{\mu2x} + p_{\gamma1x} + p_{\gamma2x})\sigma_{p_{\mu1x}E_{\mu1}}(E_{\mu2} + E_{\gamma1} + E_{\gamma2}) & +(p_{\mu2x} + p_{\gamma1x} + p_{\gamma2x})\sigma^2_{p_{\mu1x}}(p_{\mu2x} + p_{\gamma1x} + p_{\gamma2x}) \\
+ (p_{\mu2x} + p_{\gamma1x} + p_{\gamma2x})\sigma_{p_{\mu1x}p_{\mu1y}}(p_{\mu2y} + p_{\gamma1y} + p_{\gamma2y}) & +(p_{\mu2x} + p_{\gamma1x} + p_{\gamma2x})\sigma_{p_{\mu1x}p_{\mu1z}}(p_{\mu2z} + p_{\gamma1z} + p_{\gamma2z}) \\
- (p_{\mu2y} + p_{\gamma1y} + p_{\gamma2y})\sigma_{p_{\mu1y}E_{\mu1}}(E_{\mu2} + E_{\gamma1} + E_{\gamma2}) & +(p_{\mu2y} + p_{\gamma1y} + p_{\gamma2y})\sigma_{p_{\mu1y}p_{\mu1x}}(p_{\mu2x} + p_{\gamma1x} + p_{\gamma2x}) \\
+ (p_{\mu2y} + p_{\gamma1y} + p_{\gamma2y})\sigma^2_{p_{\mu1y}}(p_{\mu2y} + p_{\gamma1y} + p_{\gamma2y}) & +(p_{\mu2y} + p_{\gamma1y} + p_{\gamma2y})\sigma_{p_{\mu1y}p_{\mu1z}}(p_{\mu2z} + p_{\gamma1z} + p_{\gamma2z}) \\
- (p_{\mu2z} + p_{\gamma1z} + p_{\gamma2z})\sigma_{p_{\mu1z}E_{\mu1}}(E_{\mu1} + E_{\mu2} + E_{\gamma2}) & +(p_{\mu2z} + p_{\gamma1z} + p_{\gamma2z})\sigma_{p_{\mu1z}p_{\mu1x}}(p_{\mu2x} + p_{\gamma1x} + p_{\gamma2x}) \\
+ (p_{\mu2z} + p_{\gamma1z} + p_{\gamma2z})\sigma_{p_{\mu1z}p_{\mu1y}}(p_{\mu2y} + p_{\gamma1y} + p_{\gamma2y}) & +(p_{\mu2z} + p_{\gamma1z} + p_{\gamma2z})\sigma^2_{p_{\mu1z}}(p_{\mu2z} + p_{\gamma1z} + p_{\gamma2z}) \\
 & \\
+ (E_{\mu1} + E_{\mu2} + E_{\gamma2})\sigma^2_{E_{\gamma1}}(E_{\mu1} + E_{\mu2} + E_{\gamma2}) & -(E_{\mu1} + E_{\mu2} + E_{\gamma2})\sigma_{E_{\gamma1}p_{\gamma1x}}(p_{\mu1x} + p_{\mu2x} + p_{\gamma2x}) \\
- (E_{\mu1} + E_{\mu2} + E_{\gamma2})\sigma_{E_{\gamma1}p_{\gamma1y}}(p_{\mu1y} + p_{\mu2y} + p_{\gamma2y}) & -(E_{\mu1} + E_{\mu2} + E_{\gamma2})\sigma_{E_{\gamma1}p_{\gamma1z}}(p_{\mu1z} + p_{\mu2z} + p_{\gamma2z}) \\
- (p_{\mu1x} + p_{\mu2x} + p_{\gamma2x})\sigma_{p_{\gamma1x}E_{\gamma1}}(E_{\mu1} + E_{\mu2} + E_{\gamma2}) & +(p_{\mu1x} + p_{\mu2x} + p_{\gamma2x})\sigma^2_{p_{\gamma1x}}(p_{\mu1x} + p_{\mu2x} + p_{\gamma2x}) \\
+ (p_{\mu1x} + p_{\mu2x} + p_{\gamma2x})\sigma_{p_{\gamma1x}p_{\gamma1y}}(p_{\mu1y} + p_{\mu2y} + p_{\gamma2y}) & +(p_{\mu1x} + p_{\mu2x} + p_{\gamma2x})\sigma_{p_{\gamma1x}p_{\gamma1z}}(p_{\mu1z} + p_{\mu2z} + p_{\gamma2z}) \\
- (p_{\mu1y} + p_{\mu2y} + p_{\gamma2y})\sigma_{p_{\gamma1y}E_{\gamma1}}(E_{\mu1} + E_{\mu2} + E_{\gamma2}) & +(p_{\mu1y} + p_{\mu2y} + p_{\gamma2y})\sigma_{p_{\gamma1y}p_{\gamma1x}}(p_{\mu1x} + p_{\mu2x} + p_{\gamma2x}) \\
+ (p_{\mu1y} + p_{\mu2y} + p_{\gamma2y})\sigma^2_{p_{\gamma1y}}(p_{\mu1y} + p_{\mu2y} + p_{\gamma2y}) & +(p_{\mu1y} + p_{\mu2y} + p_{\gamma2y})\sigma_{p_{\gamma1y}p_{\gamma1z}}(p_{\mu1z} + p_{\mu2z} + p_{\gamma2z}) \\
- (p_{\mu1z} + p_{\mu2z} + p_{\gamma2z})\sigma_{p_{\gamma1z}E_{\gamma1}}(E_{\mu1} + E_{\mu2} + E_{\gamma2}) & +(p_{\mu1z} + p_{\mu2z} + p_{\gamma2z})\sigma_{p_{\gamma1z}p_{\gamma1x}}(p_{\mu1x} + p_{\mu2x} + p_{\gamma2x}) \\
+ (p_{\mu1z} + p_{\mu2z} + p_{\gamma2z})\sigma_{p_{\gamma1z}p_{\gamma1y}}(p_{\mu1y} + p_{\mu2y} + p_{\gamma2y}) & +(p_{\mu1z} + p_{\mu2z} + p_{\gamma2z})\sigma^2_{p_{\gamma1z}}(p_{\mu1z} + p_{\mu2z} + p_{\gamma2z}) \\
 & \\
+ (E_{\mu1} + E_{\gamma1} + E_{\gamma2})\sigma^2_{E_{\mu2}}(E_{\mu1} + E_{\gamma1} + E_{\gamma2}) & -(E_{\mu1} + E_{\gamma1} + E_{\gamma2})\sigma_{E_{\mu2}p_{\mu2x}}(p_{\mu1x} + p_{\gamma1x} + p_{\gamma2x}) \\
- (E_{\mu1} + E_{\gamma1} + E_{\gamma2})\sigma_{E_{\mu2}p_{\mu2y}}(p_{\mu1y} + p_{\gamma1y} + p_{\gamma2y}) & -(E_{\mu1} + E_{\gamma1} + E_{\gamma2})\sigma_{E_{\mu2}p_{\mu2z}}(p_{\mu1z} + p_{\gamma1z} + p_{\gamma2z}) \\
- (p_{\mu1x} + p_{\gamma1x} + p_{\gamma2x})\sigma_{p_{\mu2x}E_{\mu2}}(E_{\mu1} + E_{\gamma1} + E_{\gamma2}) & +(p_{\mu1x} + p_{\gamma1x} + p_{\gamma2x})\sigma^2_{p_{\mu2x}}(p_{\mu1x} + p_{\gamma1x} + p_{\gamma2x}) \\
+ (p_{\mu1x} + p_{\gamma1x} + p_{\gamma2x})\sigma_{p_{\mu2x}p_{\mu2y}}(p_{\mu1y} + p_{\gamma1y} + p_{\gamma2y}) & +(p_{\mu1x} + p_{\gamma1x} + p_{\gamma2x})\sigma_{p_{\mu2x}p_{\mu2z}}(p_{\mu1z} + p_{\gamma1z} + p_{\gamma2z}) \\
- (p_{\mu1y} + p_{\gamma1y} + p_{\gamma2y})\sigma_{p_{\mu2y}E_{\mu2}}(E_{\mu1} + E_{\gamma1} + E_{\gamma2}) & +(p_{\mu1y} + p_{\gamma1y} + p_{\gamma2y})\sigma_{p_{\mu2y}p_{\mu2x}}(p_{\mu1x} + p_{\gamma1x} + p_{\gamma2x}) \\
+ (p_{\mu1y} + p_{\gamma1y} + p_{\gamma2y})\sigma^2_{p_{\mu2y}}(p_{\mu1y} + p_{\gamma1y} + p_{\gamma2y}) & +(p_{\mu1y} + p_{\gamma1y} + p_{\gamma2y})\sigma_{p_{\mu2y}p_{\mu2z}}(p_{\mu1z} + p_{\gamma1z} + p_{\gamma2z}) \\
- (p_{\mu1z} + p_{\gamma1z} + p_{\gamma2z})\sigma_{p_{\mu2z}E_{\mu2}}(E_{\mu1} + E_{\gamma1} + E_{\gamma2}) & +(p_{\mu1z} + p_{\gamma1z} + p_{\gamma2z})\sigma_{p_{\mu2z}p_{\mu2x}}(p_{\mu1x} + p_{\gamma1x} + p_{\gamma2x})
\end{array}$$

$$
\begin{aligned}
&+ (p_{\mu 1z} + p_{\gamma 1z} + p_{\gamma 2z})\sigma_{p_{\mu 2z}p_{\mu 2y}}(p_{\mu 1y} + p_{\gamma 1y} + p_{\gamma 2y}) && +(p_{\mu 1z} + p_{\gamma 1z} + p_{\gamma 2z})\sigma^2_{p_{\mu 2z}}(p_{\mu 1z} + p_{\gamma 1z} + p_{\gamma 2z}) \\
\\
&+ (E_{\mu 1} + E_{\mu 2} + E_{\gamma 1})\sigma^2_{E_{\gamma 2}}(E_{\mu 1} + E_{\mu 2} + E_{\gamma 1}) && -(E_{\mu 1} + E_{\mu 2} + E_{\gamma 1})\sigma_{E_{\gamma 2}p_{\gamma 2x}}(p_{\mu 1x} + p_{\mu 2x} + p_{\gamma 1x}) \\
&- (E_{\mu 1} + E_{\mu 2} + E_{\gamma 1})\sigma_{E_{\gamma 2}p_{\gamma 2y}}(p_{\mu 1y} + p_{\mu 2y} + p_{\gamma 1y}) && -(E_{\mu 1} + E_{\mu 2} + E_{\gamma 1})\sigma_{E_{\gamma 2}p_{\gamma 2z}}(p_{\mu 1z} + p_{\mu 2z} + p_{\gamma 1z}) \\
&- (p_{\mu 1x} + p_{\mu 2x} + p_{\gamma 1x})\sigma_{p_{\gamma 2x}E_{\gamma 2}}(E_{\mu 1} + E_{\mu 2} + E_{\gamma 1}) && +(p_{\mu 1x} + p_{\mu 2x} + p_{\gamma 1x})\sigma^2_{p_{\gamma 2x}}(p_{\mu 1x} + p_{\mu 2x} + p_{\gamma 1x}) \\
&+ (p_{\mu 1x} + p_{\mu 2x} + p_{\gamma 1x})\sigma_{p_{\gamma 2x}p_{\gamma 2y}}(p_{\mu 1y} + p_{\mu 2y} + p_{\gamma 1y}) && +(p_{\mu 1x} + p_{\mu 2x} + p_{\gamma 1x})\sigma_{p_{\gamma 2x}p_{\gamma 2z}}(p_{\mu 1z} + p_{\mu 2z} + p_{\gamma 1z}) \\
&- (p_{\mu 1y} + p_{\mu 2y} + p_{\gamma 1y})\sigma_{p_{\gamma 2y}E_{\gamma 2}}(E_{\mu 1} + E_{\mu 2} + E_{\gamma 1}) && +(p_{\mu 1y} + p_{\mu 2y} + p_{\gamma 1y})\sigma_{p_{\gamma 2y}p_{\gamma 2x}}(p_{\mu 1x} + p_{\mu 2x} + p_{\gamma 1x}) \\
&+ (p_{\mu 1y} + p_{\mu 2y} + p_{\gamma 1y})\sigma^2_{p_{\gamma 2y}}(p_{\mu 1y} + p_{\mu 2y} + p_{\gamma 1y}) && +(p_{\mu 1y} + p_{\mu 2y} + p_{\gamma 1y})\sigma_{p_{\gamma 2y}p_{\gamma 2z}}(p_{\mu 1z} + p_{\mu 2z} + p_{\mu 1z}) \\
&- (p_{\mu 1z} + p_{\mu 2z} + p_{\gamma 1z})\sigma_{p_{\gamma 2z}E_{\gamma 2}}(E_{\mu 1} + E_{\mu 2} + E_{\gamma 1}) && +(p_{\mu 1z} + p_{\mu 2z} + p_{\gamma 1z})\sigma_{p_{\gamma 2z}p_{\gamma 2x}}(p_{\mu 1x} + p_{\mu 2x} + p_{\gamma 1x}) \\
&+ (p_{\mu 1z} + p_{\mu 2z} + p_{\gamma 1z})\sigma_{p_{\gamma 2z}p_{\gamma 2y}}(p_{\mu 1y} + p_{\mu 2y} + p_{\gamma 1y}) && +(p_{\mu 1z} + p_{\mu 2z} + p_{\gamma 1z})\sigma^2_{p_{\gamma 2z}}(p_{\mu 1z} + p_{\mu 2z} + p_{\mu 1z}) \\
&].
\end{aligned}
\tag{17}
$$

Now, considering 16 terms which only have the covariance relevant to $\mu 1$. Define variables and covariance matrix like below;

$$
E_{\neg\mu 1} \equiv E_{\mu 2} + E_{\gamma 1} + E_{\gamma 2}, \tag{18}
$$

$$
p_{\neg\mu 1\alpha} \equiv p_{\mu 2\alpha} + p_{\gamma 1\alpha} + p_{\gamma 2\alpha} \quad (\alpha = x, y, z), \tag{19}
$$

$$
P_{\neg\mu 1} \equiv \begin{pmatrix} E_{\neg\mu 1} \\ p_{\neg\mu 1x} \\ p_{\neg\mu 1y} \\ p_{\neg\mu 1z} \end{pmatrix}, \tag{20}
$$

$$
P^T_{\neg\mu 1} \equiv \begin{pmatrix} E_{\neg\mu 1} & -p_{\neg\mu 1x} & -p_{\neg\mu 1y} & -p_{\neg\mu 1z} \end{pmatrix}, \tag{21}
$$

$$
\Sigma_{\mu 1} \equiv \begin{pmatrix} \sigma^2_{E_{\mu 1}} & \sigma_{E_{\mu 1}p_{\mu 1x}} & \sigma_{E_{\mu 1}p_{\mu 1y}} & \sigma_{E_{\mu 1}p_{\mu 1z}} \\ \sigma_{p_{\mu 1x}E_{\mu 1}} & \sigma^2_{p_{\mu 1x}} & \sigma_{p_{\mu 1x}p_{\mu 1y}} & \sigma_{p_{\mu 1x}p_{\mu 1z}} \\ \sigma_{p_{\mu 1y}E_{\mu 1}} & \sigma_{p_{\mu 1y}p_{\mu 1x}} & \sigma^2_{p_{\mu 1y}} & \sigma_{p_{\mu 1y}p_{\mu 1z}} \\ \sigma_{p_{\mu 1z}E_{\mu 1}} & \sigma_{p_{\mu 1z}p_{\mu 1x}} & \sigma_{p_{\mu 1z}p_{\mu 1y}} & \sigma^2_{p_{\mu 1z}} \end{pmatrix}. \tag{22}
$$

The symbol "$\neg$" means NOT in mathematics, but in this note we will use this symbol like $E_{\neg\mu 1}$ with the meaning of "not contains $E_{\mu 1}$, but contains $E_{\mu 2} + E_{\gamma 1} + E_{\gamma 2}$" for example. We can also define the similar variables for $\mu 2$, $\gamma 1$, and $\gamma 2$. By using these definitions, finally we can write down $\sigma(M)$ as;

$$
\sigma^2(M) = \frac{1}{M^2}\left[P^T_{\neg\mu 1}\Sigma_{\mu 1}P_{\neg\mu 1} + P^T_{\neg\mu 2}\Sigma_{\mu 2}P_{\neg\mu 2} + P^T_{\neg\gamma 1}\Sigma_{\gamma 1}P_{\neg\gamma 1} + P^T_{\neg\gamma 2}\Sigma_{\gamma 2}P_{\neg\gamma 2}\right]. \tag{23}
$$